On the Naturally Induced Sources for Obstacle Scattering

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Abstract. We introduce the equivalent sources for the Helmholtz equation and establish their connections to the naturally induced sources for the sound-soft, sound-hard, and impedance obstacles for the inverse scattering problems of the Helmholtz equation. As two applications, we employ the naturally induced sources to improve the boundary integral equation formulations for the obstacle scattering problems, and develop a unified, straightforward approach to establishing boundary conditions governing the domain derivatives of scattered waves for the soft, hard, and impedance obstacles.

Key words: Naturally induced sources; obstacle scattering; domain derivatives.

1 Introduction

The subject of this paper is on the forward and inverse obstacle scattering problems for the Helmholtz equation. We will introduce the notion of naturally induced sources in the scattering by an obstacle, and use it to reformulate the standard boundary integral equations for the forward scattering problems. We also use it to establish a unified approach to the domain derivatives for the inverse obstacle scattering problems.

An equivalent source for a time harmonic wave u_0 in a domain D is made of monopoles, dipoles, or their combination on the boundary which reproduces the wave in the domain. The problems of determining the equivalent sources given u_0 is referred to as the interior (scattering) problems. There are three standard interior problems for the monopoles, dipoles, and their linear combination.

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If the domain is the support of an obstacle, of sound-soft or sound-hard or impedance type, the scattered wave can be expressed as the potential of the single or double or combined layer, respectively. These monopole, dipole, and combined sources are referred to as the naturally induced sources for the soft, hard, and impedance obstacles, respectively. Thus, for instance, double-layer potential for the exterior Neumann problem of the Helmholtz equation employs the naturally induced (dipole) sources, whereas combined potential for the exterior Dirichlet problem does not.

<u>The interior-exterior connection.</u> We will establish connections between these interior and exterior problems by identifying the naturally induced sources for the soft, hard, impedance obstacles with the equivalent sources of monopole, dipole, combined types, respectively. There are two applications of the interior-exterior connection.

<u>Reformulate scattering problems.</u> It is typical in the scattering problems that u_0 arises from known sources outside D, so both u_0 and $\partial_n u_0$ are available on the boundary. We will make use of this flexibility and employ the equivalent sources to rewrite boundary integral equations for the exterior problems. The reformulated problems are more convenient to solve, or their solution - once obtained - easier to process, than the standard approaches.

<u>Domain derivatives.</u> The calculation of domain derivative, or more generally of the Frechet derivative of the scattered wave with repect to perturbation to the bundary of the obstacle, is an essential step for the inverse obstacle problem. With the help of the equivalent sources, we will present a unified, straightforward approach to establishing boundary conditions for the domain derivatives of the scattered waves off the soft, hard, and impedance obstacles. The domain derivatives for these three scattering problems, and for the transmission problem, have already been characterized by a number of authors, initiated by the work of Kirsch [3]; see Haddar and Kress [4] for a quite complete description of the existing work.

<u>Organization of paper.</u> Section 2 provides preliminary tools. Section 3 introduces the equivalent sources and naturally induced sources, and makes the interior-exterior connection. In Section 4 we reformulate the obstacle scattering problems. Section 5 presents a straightforward, unified approach to calculating the boundary values of the domain derivatives for the soft, hard, and impedance obstacles.

2   Analytic machinery

Let D be a domain and $k > 0$ be the wave number. Throughout the paper, we work with smooth boundary ∂D; we also assume that the incident wave u_0 is generated by sources away from the boundary, so that the classical theory for layer potentials holds and that the scattering solutions are smooth. This section summarizes basic facts for layer potentials and perturbational properties of the boundary ∂D.

2.1   Jump conditions

Denote by G the fundamental solution of the Helmholtz equation

$$\Delta w + k^2\, w = 0, \quad x \in D, \tag{2.1}$$

so that

$$G(x, \xi) = \begin{cases} \frac{i}{4} H_0(k\|x - \xi\|), & \text{in } \mathbb{R}^2, \\ e^{ik|x-\xi|}/(4\pi|x-\xi|), & \text{in } \mathbb{R}^3. \end{cases} \tag{2.2}$$

Let p, q be the single and double layer potentials of smooth density σ

$$p(x) \;=\; \int_{\partial D} G(x, \xi)\, \sigma(\xi)\, dS(\xi), \tag{2.3}$$

$$q(x) \;=\; \int_{\partial D} \frac{\partial G(x, \xi)}{\partial n(\xi)}\, \sigma(\xi)\, dS(\xi). \tag{2.4}$$

We have the four well known [1,2] jump conditions across the boundary

$$p_+ - p_- = 0, \quad \partial_n q_+ - \partial_n q_- = 0, \quad q_+ - q_- = \sigma, \quad \partial_n p_+ - \partial_n p_- = -\sigma. \tag{2.5}$$

Lemma 2.1. *Let p, q be the single and double layer potentials of a smooth density σ. Let μ, λ be scalars. Then*

$$(\mu\, \partial_n + i\lambda)\, (\mu\, q + i\lambda\, p)(x^+) = (\mu\, \partial_n + i\lambda)\, (\mu\, q + i\lambda\, p)(x^-). \tag{2.6}$$

Proof. Using all four jump conditions we have

$$
\begin{aligned}
(\mu\, \partial_n + i\lambda)\, [\mu\, q + i\lambda p](x^+) &= \left[\mu^2 \partial_n q + i\mu\, \lambda \partial_n p + i\mu\, \lambda q + (i\lambda)^2 p\right](x^+) \\
&= \left[\mu^2 \partial_n q + i\mu\, \lambda\, (\partial_n p - \sigma) + i\mu\, \lambda\, (q + \sigma) + (i\lambda)^2 p\right](x^-) \\
&= \left[\mu^2\, \partial_n q + i\mu\, \lambda \partial_n p + i\mu\, \lambda q + (i\lambda)^2 p\right](x^-) \\
&= (\mu\, \partial_n + i\lambda)\, [\mu\, q + i\lambda p](x^-),
\end{aligned}
$$

where in the second step we have used (2.5). This completes the proof of this lemma. □

2.2   Surface perturbation

Let $\xi = x(0,0)$ be a point on the smooth surface $\Gamma = \{x(s,t)\}$ locally parameterized by the arclength s, t of the two normal sections at ξ along the two principal directions τ_1, τ_2. Denote by κ_1, κ_2 the curvatures at ξ of the two sections; thus $\kappa = (\kappa_1 + \kappa_2)/2$ is the mean curvature. We orient τ_1, τ_2, and the unit normal n of Γ at ξ so that

$$n = \tau_1 \times \tau_2, \quad [\tau_1,\ \tau_2] = [x_s,\ x_t], \quad [n_s,\ n_t] = [\kappa_1 \tau_1,\ \kappa_2 \tau_2]. \tag{2.7}$$

Given a smooth function $h : \Gamma \mapsto \mathbb{R}^1$ and a real number ε, we perturb Γ by εh in the normal direction to obtain $\Gamma_\varepsilon = \{x_\varepsilon = x(s,t) + \varepsilon h(s,t)\, n(s,t)\}$. Straightforward calculations along the principal directions yield

Lemma 2.2. *Let $dS = ds\,dt$ be the surface area element at a point $\xi \in \Gamma$. Let $dS_\varepsilon = J_\varepsilon\,ds\,dt$ be the surface area element at the corresponding point $\xi_\varepsilon = \xi + \varepsilon h\,n$ on the perturbed surface Γ_ε. Then*

$$[\partial_s x_\varepsilon, \partial_t x_\varepsilon] = [\,(1+\varepsilon h\kappa_1)\tau_1 + \varepsilon h_s n,\ (1+\varepsilon h\kappa_2)\tau_2 + \varepsilon h_t n\,], \tag{2.8}$$

$$\partial_s x_\varepsilon \times \partial_t x_\varepsilon = (1+2\varepsilon h\,\kappa)n - \varepsilon h_s \tau_1 - \varepsilon h_t \tau_2 + \mathcal{O}(\varepsilon^2), \tag{2.9}$$

$$J_\varepsilon(\xi) = |\partial_s x_\varepsilon \times \partial_t x_\varepsilon| = (1+2\varepsilon h\,\kappa) + \mathcal{O}(\varepsilon^2), \tag{2.10}$$

$$J'(h;\xi) =: \left.\frac{dJ_\varepsilon(\xi)}{d\varepsilon}\right|_{\varepsilon=0} = 2\kappa h, \tag{2.11}$$

$$n'(h;\xi) =: \left.\frac{dn_\varepsilon(\xi)}{d\varepsilon}\right|_{\varepsilon=0} = -h_s \tau_1 - h_t \tau_2. \tag{2.12}$$

Lemma 2.3. *Let $\breve{\nabla}$ be the surface gradient opertor on Γ. Then the Helmholtz equation (2.1) becomes*

$$\left[\partial_n^2 + 2\kappa\partial_n + \breve{\nabla}^2 + k^2\right]w = 0. \tag{2.13}$$

`Sketch of the proof:` Parameterizing the neighborhood of the surface by

$$y(r,s,t) = x(s,t) + r\,n(s,t), \quad x \in \partial D, \tag{2.14}$$

we rewrite the Laplacian in the orthogonal curvilinear coordinates (r,s,t)

$$\partial_r^2 + 2\kappa\partial_r + \partial_s^2 + \partial_t^2 \tag{2.15}$$

3   The equivalent sources

We introduce the equivalent and naturally induced sources, and use them to establish connections between the interior and exterior scattering problems.

3.1   The interior problems: equivalent sources

Let us refer to the sources outside D, which generate the incident wave u_0, as the primary sources. As is well known, it is possible to use equivalent sources on ∂D to reproduce the same incident wave u_0 inside D; the Green's representation theorem

$$u_0(x) = \int_{\partial D} \left(\partial_n u_0(\xi)\,G(x,\xi) - u_0(\xi)\frac{\partial G(x,\xi)}{\partial n(\xi)}\right) dS(\xi) \tag{3.1}$$

for example, does it with monopole density $\partial_n u_0$ and dipole density u_0 on ∂D.

We introduce three interior problems as to determine the equivalent sources α, β, and γ such that their corresponding single, double, and combined layer potentials all match

u_0 inside D

$$u_0(x) \; = \; \int_{\partial D} G(x,\xi)\,\alpha(\xi)\,dS, \qquad\qquad x \in D, \qquad\qquad (3.2)$$

$$u_0(x) \; = \; \int_{\partial D} \frac{\partial G(x,\xi)}{\partial n(\xi)}\,\beta(\xi)\,dS, \qquad\qquad x \in D, \qquad\qquad (3.3)$$

$$u_0(x) \; = \; \int_{\partial D} \left[\frac{\partial G(x,\xi)}{\partial n(\xi)} + i\lambda G(x,\xi)\right]\gamma(\xi)\,dS, \quad x \in D, \qquad (3.4)$$

with $\lambda \neq 0$ a real number. As is well known, the three problems have unique and smooth solutions, see [1], or see the proof of Theorem 3.1.

3.2   The exterior-interior connection

Let D be the domain of the obstacle, and u_0 be the incident wave. Let v_j and $u_j = u_0 + v_j$, $j = 1, 2, 3$ be the scattered and total waves for (i) sound-soft (ii) sound-hard (iii) impedance problems:

$$\Delta v_j(x) + k^2 v_j(x) = 0, \quad x \in R^3 \setminus \bar{D}, \qquad\qquad (3.5)$$

$$u_1 = 0, \quad \partial_n u_2 = 0, \quad (\partial_n + i\lambda)u_3 = 0, \quad x \in \partial D. \qquad (3.6)$$

Together with the Sommerfeld radiation condition at the infinity, these boundary value problems are well-posed for real numbers $k > 0$, $\lambda \neq 0$. It follows immediately from the Green's representation theorem

$$v(x) = -\int_{\partial D}\left(\partial_n u(\xi)\,G(x,\xi) - u(\xi)\frac{\partial G(x,\xi)}{\partial n(\xi)}\right)dS(\xi) \qquad (3.7)$$

and [1], Section 3.7, that the scattered waves can be expressed as the layer potentials of single, double, and combined types

$$v_1(x) \; = \; \int_{\partial D} G(x,\xi)\,a(\xi)\,dS, \qquad\qquad (3.8)$$

$$v_2(x) \; = \; \int_{\partial D} \frac{\partial G(x,\xi)}{\partial n(\xi)}\,b(\xi)\,dS, \qquad\qquad (3.9)$$

$$v_3(x) \; = \; \int_{\partial D} \left[\frac{\partial G(x,\xi)}{\partial n(\xi)} + i\lambda G(x,\xi)\right]c(\xi)\,dS, \qquad (3.10)$$

and that the densities are

$$a = -\partial_n u_1, \quad b = u_2, \quad c = u_3. \qquad\qquad (3.11)$$

Definition 3.1. The densities a, b, c are referred to as the naturally induced sources for the sound-soft, sound-hard, and impedance obstacles, respectively.

The three exterior obstacle scattering problems of determining the naturally induced sources a, b, c are identical (up to a sign) to the three interior problems of determining the monopole, dipole, and combined equivalent sources.

Theorem 3.1. *Let $k > 0$, $\lambda \neq 0$ be real numbers, and ∂D be smooth. Let α, β, and γ be the equivalent sources of monopole, dipole, and combined types for the incident wave u_0. Let a, b, c be the naturally induced sources. Then*

$$a = -\alpha, \quad b = -\beta, \quad c = -\gamma. \tag{3.12}$$

Proof. These are direct consequences of the jump conditions (2.5): v_1, $\partial_n v_2$, and $(\partial_n + i\lambda)v_3$ are all continuous across ∂D. To illustrate, we only prove the last one, and it remains to verify the impedance boundary condition for v. Indeed,

$$
\begin{aligned}
&(\partial_n + i\lambda)\, v_3(x^+) \\
=\ & -(\partial_n + i\lambda) \int_{\partial D} \left[\frac{\partial G(x^+, \xi)}{\partial n(\xi)} + i\lambda G(x^+, \xi) \right] \gamma(\xi)\, dS \\
=\ & -(\partial_n + i\lambda) \int_{\partial D} \left[\frac{\partial G(x^-, \xi)}{\partial n(\xi)} + i\lambda G(x^-, \xi) \right] \gamma(\xi)\, dS \\
=\ & -(\partial_n + i\lambda)\, u_0(x^-) = -(\partial_n + i\lambda)\, u_0(x),
\end{aligned}
$$

where in the second step we have used (2.6). This completes the proof of this theorem. $\square$

4 Equivalent source method for obstacle scattering

We will use the equivalent sources to rewrite boundary integral equations for obstacle scattering. The reformulated problems are more convenient to solve, or their solution easier to process, than the standard approaches. The integral equations we present here are not structurally new, but they explore the flexibility in reinterpreting the incident wave as the boundary data to rearrange the solution process.

Therefore and again, we assume that the incident wave u_0 is generated by known sources away from the boundary, so that both u_0 and $\partial_n u_0$ can be evaluated on the boundary. For simplicity, we assume that $k > 0$, $\lambda \neq 0$, $\mu \neq 0$ are real numbers.

4.1 sound-soft problem

The problem is to determine density a of (3.8), which according to Theorem 3.1 is to determine the equivalent monopole source α of (3.2). Applying $\partial_n + i\mu$ to (3.2) from the interior side of ∂D we obtain the second kind integral equation for α

$$
\alpha(x)/2 + \int_{\partial D} \left[\frac{\partial G(x, \xi)}{\partial n(x)} + i\mu G(x, \xi) \right] \alpha(\xi)\, dS = h(x),
$$
$$
\text{with} \quad h = (\partial_n + i\mu)u_0, \tag{4.1}
$$

or $(I + K' + i\mu S)\alpha = 2h$ in the standard operator form - the adjoint of the well known combined potential equation, where μ is the combination coefficient of the single and double layer potentials. A nonzero, real μ makes (4.1) uniquely solvable [1]. This equation was previously used by Burton and Miller [5]. Thus a second kind integral equation, free of resonances, has been gained in spite of our using single layer potential. The scattered wave (3.8) is easier to process, such as taking derivative on or near the boundary numerically, than the standard combined potential approach to the sound-soft problem.

4.2   Sound-hard and impedance problems

In the following treatment, the sound-hard problem is a special case of the impedance problem with $\lambda = 0$; we will thus only consider the latter.

We restrict x on ∂D in (3.4), and obtain the second kind equation for γ

$$-\gamma(x)/2 + \int_{\partial D} \left[\frac{\partial G(x,\xi)}{\partial n(\xi)} + i\lambda G(x,\xi) \right] \gamma(\xi)\,dS = u_0(x), \qquad (4.2)$$

or $(-I + K + i\lambda S)\gamma = 2u_0$, which is the adjoint of the single-layer approach to the impedance problem. Since the equation always has a solution $\gamma = u_3$, resonances occur if and only if k is an interior Dirichlet eigenvalue.

The advantage, in the absence of resonance, is obvious: the impedance condition is treated as if the Dirichlet condition is typically treated with the combined potential equation; no hypersingularity, no regularization required.

To remove the spurious modes, whose number is finite, we may choose the same number of interior points $\{x_j,\ j = 1 : p\}$, and augment (4.2) with the p additional equations

$$\int_{\partial D} \left[\frac{\partial G(x_j,\xi)}{\partial n(\xi)} + i\lambda G(x_j,\xi) \right] \gamma(\xi)\,dS = u_0(x_j) \qquad (4.3)$$

each arising from a monopole source at x_j inside D. In practice, some fixed number of interior points will be employed, whether or not k is an interior Dirichlet eigenvalue. The resulting over determined, but consistent, linear system can be solved directly with QR factorization, or iteratively with conjugate gradient method for the normal equations.

The augmenting equations (4.3) are very similar to the null field method, except that here we do not put all the points in the interior, so that the equations (4.2) and (4.3) are still of the second kind.

5   Domain derivative of scattered waves

We present a unified, straightforward approach to deriving boundary conditions for the domain derivatives of the scattered waves off the soft, hard, and impedance obstacles. As noted in Section 1, these derivatives have already been characterized, see [4] for an extensive list of existing work.

In agreement with Section 2.2, we denote by $\partial D_\varepsilon = \{\, x_\varepsilon = x + \varepsilon h\, n \,|\, x \in \partial D \,\}$ the perturbed boundary, by v_ε the perturbed scattered waves, and so on, so that the perturbed (3.8)-(3.10) can be expressed by the single formula

$$v_\varepsilon(x) = \int_{\partial D} \left[\mu\, n_\varepsilon(\xi) \cdot \nabla_{\xi_\varepsilon} G(x, \xi_\varepsilon) + i\lambda G(x, \xi_\varepsilon)\right]\sigma_\varepsilon(\xi)\, J_\varepsilon(\xi)\, dS =: M_\varepsilon(\sigma_\varepsilon) \qquad (5.1)$$

for the three cases

$$v = v_1,\ \sigma = -\partial_n u_1, \qquad \mu = 0,\ \lambda = -i \quad \text{(soft)}, \qquad\qquad (5.2)$$
$$v = v_2,\ \sigma = u_2, \qquad\quad \mu = 1,\ \lambda = 0 \qquad \text{(hard)}, \qquad\qquad (5.3)$$
$$v = v_3,\ \sigma = u_3, \qquad\quad \mu = 1,\ \lambda \in \mathbb{R}^1 \quad \text{(impedance)}. \qquad (5.4)$$

Assuming their existence and smoothness, we now derive boundary conditions for the domain derivatives

$$v'(h; x) =: \left.\frac{dv_\varepsilon(x)}{d\varepsilon}\right|_{\varepsilon=0} \qquad\qquad (5.5)$$

of scattered waves v_1, v_2, v_3 for the sound-soft, sound-hard, and impedance obstacles.

Theorem 5.1. *Let $k > 0$, $\lambda \neq 0$ be real numbers. Let ∂D and $h : \partial D \mapsto \mathbb{R}^1$ be smooth. Let $u_j = u_0 + v_j$, $j = 1, 2, 3$, be the total waves of the sound-soft, sound-hard, impedance obstacle D. Then the domain derivatives $v_j'(h; \cdot)$, should they exist and be smooth, are themselves scattered waves off the obstacle D and subject to the boundary conditions*

$$v_1'(h; x) \ =\ -h(x)\, \partial_n u_1(x), \qquad\qquad (5.6)$$
$$\partial_n v_2'(h; x) \ =\ \breve{\nabla} \cdot \left[h(x)\breve{\nabla} u_2(x)\right] + k^2 h(x)\, u_2(x), \qquad\qquad (5.7)$$
$$(\partial_n + i\lambda)v_3'(h; x) \ =\ \breve{\nabla} \cdot \left[h(x)\breve{\nabla} u_3(x)\right] + \left(k^2 - \lambda^2 - 2i\lambda\kappa\right)h(x)\, u_3(x), \qquad (5.8)$$

where $\breve{\nabla}$ is the surface gradient.

Proof. For $x \notin \partial D$, differentiate (5.1) with the chain rule to obtain

$$\begin{aligned}
v'(h; x) &= \left.\frac{dM_\varepsilon(c_\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} \\
&= \int_{\partial D} \sigma(\xi)\big\{\mu n'(h; \xi) \cdot \nabla_\xi + \mu h\, \partial_\nu^2 + i\lambda h\, \partial_\nu\big\}G(x, \xi)\, dS + M_0\big(\sigma' + \sigma\, J'\big),
\end{aligned}$$

where ν denotes the normal $n(\xi)$, as opposed to $n(x)$. Using (2.12) and (2.13) to process the first two terms, and integrating by parts using the Gauss surface divergence theorem,

we obtain

$$\int_{\partial D} \sigma(\xi)\{n'(h;\xi)\cdot\nabla_\xi + h\,\partial_\nu^2\}G(x,\xi)\,dS$$

$$= \int_{\partial D} \sigma(\xi)\{-\breve{\nabla}h\cdot\breve{\nabla}_\xi - h\big(2\kappa\partial_\nu + \breve{\nabla}_\xi^2 + k^2\big)\}G(x,\xi)\,dS$$

$$= \int_{\partial D}\big\{\breve{\nabla}\cdot\big(\sigma\breve{\nabla}h\big) - 2h\sigma\,\kappa\partial_\nu - \big(\breve{\nabla}^2 + k^2\big)h\sigma\big\}G(x,\xi)dS$$

$$= -\int_{\partial D}\big\{\breve{\nabla}\cdot\big(h\breve{\nabla}\sigma\big) + k^2 h\sigma + 2h\sigma\,\kappa\partial_\nu\big\}G(x,\xi)dS,$$

where in the second step we have used (2.12)-(2.13) and in the third step we have used the Gauss theorem. Thus,

$$v'(h;x) = M_0\big(\sigma' + \sigma\,J'\big) + \int_{\partial D}\Big\{(i\lambda - 2\mu\kappa)h\sigma\partial_\nu G - \mu\big[\breve{\nabla}\cdot\big(h\breve{\nabla}\sigma\big) + k^2 h\sigma\big]G\Big\}dS.$$

Observe that $v'(h;x) = 0$, $x\in D$, since $v(x) = v_\varepsilon(x) = -u_0(x)$, $x\in D\cap D_\varepsilon$. Now bring x back to the boundary, and use jump conditions to simplify; thus for $x\in\partial D$

$$v'(h;x^+) = v'(h;x^+) - v'(h;x^-) =: \big\lceil v'(h;x)\big\rceil = \big\lceil M_0\big(\sigma' + \sigma\,J'\big)\big\rceil +$$

$$+ \Big\lceil\int_{\partial D}\Big\{(i\lambda - 2\mu\kappa)\,h\sigma\,\partial_\nu G - \mu G\big[\breve{\nabla}\cdot\big(h\breve{\nabla}\sigma\big) + k^2 h\sigma\big]\Big\}dS\Big\rceil.$$

By taking the normal derivative on the boundary we have

$$(\mu\,\partial_n + i\lambda)v'(h;x^+) = \Big[(\mu\,\partial_n + i\lambda)M_0\big(\sigma' + \sigma\,J'\big)\Big] +$$

$$+ \Big[(\mu\,\partial_n + i\lambda)\int_{\partial D}\Big\{(i\lambda - 2\mu\,\kappa)\,h\sigma\,\partial_\nu G - \mu\,G\big[\breve{\nabla}\cdot\big(h\breve{\nabla}\sigma\big) + k^2 h\sigma\big]\Big\}dS\Big].$$

The first part vanishes since $(\mu\,\partial_n + i\lambda)M_0$ is continuous across the boundary due to Lemma 2.1. The second part has four terms, of which only two jump across the boundary

$$(\mu\,\partial_n + i\lambda)v'(h;x^+)$$

$$= \Big[\int_{\partial D}\Big\{i\lambda(i\lambda - 2\mu\,\kappa)\,h\sigma\,\frac{\partial G(x,\xi)}{\partial n(\xi)} - \mu^2\big[\breve{\nabla}\cdot\big(h\breve{\nabla}\sigma\big) + k^2 h\sigma\big]\frac{\partial G(x,\xi)}{\partial n(x)}\Big\}dS\Big]$$

$$= \mu^2\,\breve{\nabla}\cdot\big(h\breve{\nabla}\sigma\big) + (\mu^2\,k^2 - \lambda^2 - 2i\mu\,\lambda\kappa)\,h\sigma.$$

Now (5.6)-(5.8) follow immediately from (5.2)-(5.4)   □

In the proof the smoothness of σ' is assumed, which is valid because

$$\frac{dv_\varepsilon(x_\varepsilon)}{d\varepsilon}\Big|_{\varepsilon=0} = v'(h;x) + h\partial_n v(x) \qquad\qquad (5.9)$$

$$\frac{d\partial_n v_\varepsilon(x_\varepsilon)}{d\varepsilon}\Big|_{\varepsilon=0} = \Lambda\Big[\frac{dv_\varepsilon(x_\varepsilon)}{d\varepsilon}\Big]_{\varepsilon=0} - \breve{\nabla}v\cdot\breve{\nabla}h, \qquad\qquad (5.10)$$

where $\Lambda : C(\partial D)\mapsto C(\partial D)$ is the Dirichlet-to-Neumann map.

Remark 5.1. By Theorem 5.1, the domain derivatives $v'_j(h; \cdot)$ can be obtained as solutions of boundary integral equations for the sound-soft, sound-hard, impedance scattering problems for the obstacle D; see, e.g., [1] for more details on these boundary integral equations.

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